Classification of Associative Two-Dimensional Algebras

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Abstract Algebra

We normally work with a one-dimensional algebra. For example,

\[ 3 = 3(1) \]
\[ 10 = 10(1) \]
\[ -5 = -5(1) \]

Starting with the unit vector, we can obtain any quantity we are looking for by multiplying by some scalar in \( \mathbb{R} \).
Two-Dimensional Algebras

Consider a negative real number, say $-1$. We may want $x$ such that $x^2 = -1$.

But our one-dimensional algebra cannot handle this situation. So we use a second dimension in the $i$ direction.

http://pirate.shu.edu/ wachsmut/complex/numbers/graphics/plane.gif
Describing an Algebra

Multiplication tables describe an algebra visually. They are read much like Punnett squares in biology.

Example: The complex number system

\[
\begin{array}{c|cc}
\times & 1 & i \\
\hline 
1 & 1 & i \\
i & i & -1 \\
\end{array}
\]

\[
\begin{array}{c|cc}
\times & u & v \\
\hline 
u & u & v \\
v & v & -u \\
\end{array}
\]
An algebra $\mathcal{A}$ is

- **associative** if and only if $(xy)z = x(yz)$ for all $x, y, z \in \mathcal{A}$
- **commutative** if and only if $xy = yx$ for all $x, y \in \mathcal{A}$
- **unital** if and only if $\exists u \in \mathcal{A} | ux = x$ for all $x \in \mathcal{A}$
- **a division algebra** if and only if for every $x \in \mathcal{A}$ there exists $x' \in \mathcal{A} | xx' = x'x = 1$.

For the purposes of this investigation, we only assume associativity.
Properties of Two-Dimensional Algebras

Any two algebras within a given category may be isomorphic.
Isomorphism

From Greek *iso*, meaning “equal,” and *morphosis*, meaning “to form”

Example: Two isomorphic algebras

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & 0 \\
v & v & 0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & u \\
v & v & v \\
\end{array}
\]
Isomorphism (continued)

Why are these two algebras isomorphic?

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & 0 \\
v & v & 0 \\
\end{array}
\quad
\begin{array}{c|ccc}
\times & u & w \\
\hline
u & u & u \\
w & w & w \\
\end{array}
\]

Start with the first algebra. Then choose \( w := u + \beta v \).

\[
\begin{align*}
\text{\( uu \)} & \text{ does not change} \\
\text{\( uw = uu + \beta uv = uu = u \)} \\
\text{\( wu = uu + \beta vu = u + \beta v = w \)} \\
\text{\( ww = (u + \beta v)(u + \beta v) = uu + \beta vu = u + \beta v = w \)}
\end{align*}
\]

So there exists a mapping between these two algebras.
The field $\mathbb{F}$ is a one-dimensional, unital, and commutative algebra that includes all the allowable coefficients for our two-dimensional algebras.

We assume the field to be of characteristic 0.

For non-unital algebras, there are no other restrictions on the field.

However, for division algebras, we assume the field to be $\mathbb{R}$. 
A Useful Lemma

**Lemma 1.**

*For an algebra \( A \) there exists \( z \in A^\times \) such that either \( zz = 0 \) or \( zz = z \).*

**Proof.**

Consider any \( u \in A^\times \).

- \( uu \in F \cdot u \). Let \( uu = ru \) for \( r \in F \). If \( r = 0 \), then \( uu = 0 \). Otherwise, define \( v := \frac{1}{r} u \). \( vv = \frac{1}{r^2} uu = \frac{1}{r} u = v \).

- Case 2: \( u \) and \( uu \) are not collinear. Thus \((u, uu)\) is a basis. Suppose \( uuu = \alpha u + \beta uu \) for \( \alpha, \beta \in F \), and choose \( y := uu - \beta u \). Then we have \( yy = \alpha y \). We have now reduced the problem to a form covered by Case 1, for which we have already shown the conclusion holds.
Classifying the Non-unital Algebras

Assume $\exists u \in A \mid uu = u$.

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & \alpha u + \beta v \\
v & \gamma u + \delta v & \\
\end{array}
\]

$u(uv) = u(\alpha u + \beta v) = \alpha uu + \beta uv = \alpha u + \beta(\alpha u + \beta v) = (\alpha + \alpha\beta)u + \beta^2 v$

$\triangleright (uu)v = uv = \alpha u + \beta v$
Classifying the Non-unital Algebras (continued)

By the Associative Law,

$$u(uv) = (uu)v$$

$$(\alpha + \alpha \beta)u + \beta^2 v = \alpha u + \beta v$$

Matching the coefficients gives us a system of equalities:

$$\begin{cases} 
\alpha + \alpha \beta = \alpha \\
\beta^2 = \beta 
\end{cases} \quad \text{therefore} \quad \begin{cases} 
\alpha \beta = 0 \\
\beta(\beta - 1) = 0 
\end{cases}$$
The Associative Law requires that

\[
\begin{align*}
\alpha\beta &= 0 \\
\gamma\delta &= 0 \\
\beta(\beta - 1) &= 0 \\
\delta(\delta - 1) &= 0 \\
\alpha(\delta - 1) &= \gamma(\beta - 1)
\end{align*}
\]

Thus there are four cases:

1. \( \beta = \delta = 0 \)
2. \( \beta = 1, \delta = 0 \)
3. \( \beta = 0, \delta = 1 \)
4. \( \beta = \delta = 1 \)
Case 1: $\beta = \delta = 0$

\[
\alpha(\delta - 1) = \gamma(\beta - 1) \implies \gamma = \alpha
\]

| $\times$ | $u$ | $v$
|---------|----|----|
| $u$     | $u$ | $\alpha u$
| $v$     | $\alpha u$ |

We can simplify this algebra by assuming $\alpha \neq 0$ and choosing $w := u - \frac{1}{\alpha} v$.

\[
uw = u \left( u - \frac{1}{\alpha} v \right) = u - u = 0
\]

\[
wu = \left( u - \frac{1}{\alpha} v \right) u = u - u = 0
\]
Case 1: $\beta = \delta = 0$ (continued)

Let $ww := \epsilon u + \zeta w$. By the Associative Law,

$$(\epsilon + \zeta w)u = (ww)u = w(wu) = 0 \implies \epsilon = 0$$

and thus $ww = \zeta w$.

- If $\zeta \neq 0$, choose $v' := \frac{1}{\zeta} w$ which gives $v'v' = \frac{1}{\zeta^2} ww = \frac{1}{\zeta} w = v'$. Next, choose $z := u + v'$. If $(z, v')$ is the basis, we obtain a unital case. This is isomorphic to the $\alpha = 0$ case.

\[
\begin{array}{c|c|c}
\times & z & v' \\
\hline
z & z & v' \\
v' & v' & v' \\
\end{array}
\]

- If $\zeta = 0$, we obtain a non-unital case.

\[
\begin{array}{c|c|c}
\times & u & w \\
\hline
u & u & 0 \\
w & 0 & 0 \\
\end{array}
\]
The Five Non-unital Algebras

The algebra we just found is denoted by I. Cases 2, 3, and 4 yield II and III, and the \( \nexists u \in A^\times \mid uu = u \) scenario yields IV and V.

<table>
<thead>
<tr>
<th></th>
<th>uu</th>
<th>uv</th>
<th>vu</th>
<th>vv</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>u</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>u</td>
<td>v</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>III</td>
<td>u</td>
<td>0</td>
<td>v</td>
<td>0</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>u</td>
</tr>
<tr>
<td>V</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Some important observations:

- I, IV, and V are commutative while II and III are not.
- None of these five algebras are isomorphic. Hence, we have a complete classification of the non-unital algebras.
Unital Algebras

An algebra is unital if there exists a *unity* $u$ such that $uv = v = vu$ for all $v$. A unital algebra is *divisional* if and only if there exists $y$ for each $x$ such that $yx = u$.

**Lemma 2.**

*For all unital algebras $A$ where $u$ is the unity,\[
\exists w \in A \setminus F u \mid ww = ru \text{ for suitable } r \in F.\]*

**Proof.**

If $(u, v)$ is the basis, choose

- $r := \alpha + \frac{1}{4} \beta^2$
- $w := v - \frac{1}{2} \beta v$
- $vv := \alpha u + \beta v$

Then verify that $ww = ru$. □
Unital Algebras (continued)

**Theorem 3.**

Consider a unital algebra $A$ with unity $u$ and a vector $v$ such that $vv = ru$. $A$ is a division algebra if and only if $r \neq s^2 \ \forall s \in F$.

**Proof.**

We choose $x := \alpha u + \beta v \in A^\times$.

1. If $r \neq s^2$, then $\alpha^2 - r\beta^2 \neq 0$. Choose $y := \frac{1}{\alpha^2 - r\beta^2}(\alpha u - \beta v)$. Then $xy = yx = u$ and so the algebra is a division algebra.

2. If $A$ is divisional, $y$ (the inverse of $x$) exists for all $\alpha$ and $\beta$, and thus $\alpha^2 - r\beta^2 \neq 0 \implies r \neq (\alpha/\beta)^2$. \qed
For the division algebras, we specify that the field is $\mathbb{R}$.

1. $r = 0$: not a division algebra

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & v \\
v & v & 0
\end{array}
\]

2. $r = 1$: not a division algebra

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & v \\
v & v & u
\end{array}
\]

3. $r = -1$: a division algebra

\[
\begin{array}{c|cc}
\times & u & v \\
\hline
u & u & v \\
v & v & -u
\end{array}
\]
The Eight Two-Dimensional Associative Algebras

**Theorem 4 (Main result).**

*Every two-dimensional associative algebra is either equivalent or isomorphic to one of the eight algebras specified below.*

<table>
<thead>
<tr>
<th></th>
<th>( uu )</th>
<th>( uv )</th>
<th>( vu )</th>
<th>( vv )</th>
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</thead>
<tbody>
<tr>
<td>I</td>
<td>( u )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>( u )</td>
<td>( v )</td>
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<tr>
<td>III</td>
<td>( u )</td>
<td>0</td>
<td>( v )</td>
<td>0</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( u )</td>
</tr>
<tr>
<td>V</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>VI</td>
<td>( u )</td>
<td>( v )</td>
<td>( v )</td>
<td>0</td>
</tr>
<tr>
<td>VII</td>
<td>( u )</td>
<td>( v )</td>
<td>( v )</td>
<td>( u )</td>
</tr>
<tr>
<td>VIII</td>
<td>( u )</td>
<td>( v )</td>
<td>( v )</td>
<td>( -u )</td>
</tr>
</tbody>
</table>
Further Research

Retracting any one of the fundamental assumptions made in this investigation changes the problem significantly.

Future research could entail classification of algebras that are

- of three or more dimensions
- nonassociative
- over a field not of characteristic 0

Additionally, there is more work to be done with the two-dimensional associative division algebras.
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Thank you. Any questions?
Images
http://pirate.shu.edu/ wachsmut/complex/numbers/graphics/plane.gif
http://i.imgur.com/F6N5EYC.png